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LETTER TO THE EDITOR

An inequality for the sum of entropies of unbiased quantum measurements

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Abstract. In this letter an inequality, similar to Kraus's proposal proved by Maassen and Uffink, concerning the lower limit for the sum of entropies for unbiased quantum measurements is given. We show that $\Sigma_{i=1}^{N+1} S(W^{(i)}) \geqslant (N+1)\ln((N+1)/2)$ where $W^{(i)}$ are results of $N+1$ mutually unbiased measurements performed on the same initial state.

In a recent letter [1] it was shown that for two unbiased orthogonal ray resolutions of the identity (ORRI) in an N-dimensional complex space, $\{P_k\}$ and $\{Q_r\}$ and for any initial state W, ($W \geqslant 0$, tr $W = 1$) the following inequality is valid

$$S(W^{(1)}) + S(W^{(2)}) \geqslant \ln N \tag{1}$$

where $W^{(1)} = \Sigma_k P_k W P_k$, $W^{(2)} = \Sigma_r Q_r W Q_r$ and $S(W) = -\mathrm{tr}(W \ln W)$. We assume that two ORRIs, $\{P_k\}$ and $\{Q_r\}$ are unbiased if

$$\mathrm{tr}(P_k Q_r) = \frac{1}{N} \qquad \forall k, r.$$

Inequality (1), suggested by Kraus [2], is the strongest possible. In this letter we will give an inequality which is stronger when more unbiased measurements are possible. In particular, for $N = p^s$ where p is a prime number, $(N+1)$ such mutually unbiased ORRIs do exist [3]. Furthermore, in the space of all operators acting over C^N two unbiased ORRIs define two subspaces which are, with the exception of their common element (the identity operator), mutually orthogonal. The change of state in the standard quantum measurement, given by the 'projection postulate' is also an orthogonal projection of the initial state on the subspace defined by the ORRI corresponding to the measured observable.

With this in mind every state W can be orthogonally decomposed as

$$W = (1/N)I + \bar{W} = (1/N)I + \sum_r \bar{W}^{(r)}$$

when $(N+1)$ unbiased ORRIs are known and where $W^{(r)}$ is projection of the state W to the rth ORRI (cf [4]).

By the *length* of an operator A we shall mean its Hilbert–Schmidt norm, defined by

$$\|A\| \equiv \sqrt{\mathrm{tr}(A^\dagger A)}.$$

One easily checks that

$$\frac{1}{N} \leqslant \|W\|^2 \leqslant 1$$

and hence

$$\sum_{r=1}^{N+1} \| W^{(r)}\|^2 = \sum_{r=1}^{N+1} \left(\frac{1}{N} + \| \bar{W}^{(r)}\|^2\right) = 1 + \| W\|^2 \leq 2. \tag{2}$$

To connect the entropy to the length of a state we will use the following inequality

$$S(W) \geq -\ln \| W\|^2. \tag{3}$$

A simple proof of (3) is the following: let $\{w_i\}_{i=1}^N$ be the eigenvalues of W and define $c_i = w_i^2/\| W\|^2$. Then from

$$2S(W) = -\ln \| W\|^2 - \sum_i w_i \ln(c_i)$$

with the use of $S(W) \leq -\Sigma_i w_i \ln(c_i)$ (inequality for relative entropy) one obtains (3).

A consequence is that

$$\sum_{r=1}^{N+1} S(W^{(r)}) \geq - \sum_{r=1}^{N+1} \ln \| W^{(r)}\|^2.$$

By the concavity of the logarithm and relation (2) we now have

$$\frac{1}{N+1} \sum_{r=1}^{n+1} \ln \| W^{(r)}\|^2 \leq \ln\left(\frac{1}{N+1} \sum_{r=1}^{N+1} \| W^{(r)}\|^2\right) \leq \ln \frac{2}{N+1}$$

so that we may conclude

$$\sum_{r=1}^{N+1} S(W^{(r)}) \geq (N+1) \ln\left(\frac{N+1}{2}\right). \tag{4}$$

One should notice that an unmodified application of (1) to this case gives

$$\sum_{r=1}^{N+1} S(W^{(r)}) \geq \left(\frac{N+1}{2}\right) \ln N \tag{5}$$

and for $N \geq 4$ inequality (4) becomes stronger than (5). Obviously, inequality (1) was not intended for this case, but bearing in mind its quality it will be very interesting to find a similar technique for $N+1$ unbiased measurements.

By way of conclusion a remark can be made on a possible improvement on the lower bound for (4). It should follow either from the use of an inequality stronger than (3) or from a better insight into the set of states and its properties. For example, there is no *a priori* reason to assume that a lower bound in (3) can be reached for every N. What is certain, at least, is the simplicity of the new inequality.

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